

FLEXIBILITY OF AFFINE CONES OVER DEL PEZZO SURFACES OF DEGREE 4 AND 5

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ABSTRACT. We prove that the action of the special automorphism group on affine cones over del Pezzo surfaces of degree 4 and 5 is infinitely transitive.

1. INTRODUCTION

An affine algebraic variety X defined over an algebraically closed field \mathbb{K} of characteristic zero is called *flexible* if the tangent space of X at any smooth point is spanned by the tangent vectors to the orbits of one-parameter unipotent group actions [1]. In this paper we establish flexibility of affine cones over del Pezzo surfaces of degree 4 and 5.

It is well known that every effective action of one-dimensional unipotent group $\mathbb{G}_a = \mathbb{G}_a(\mathbb{K})$ on X defines a locally nilpotent derivation $\delta \in \text{LND}(\mathbb{K}[X])$ of the algebra of regular functions on X . All such actions generate a subgroup of *special automorphisms* $\text{SAut } X$ in the automorphism group $\text{Aut } X$.

A group G is said to act on a set S *infinitely transitively* if it acts transitively on the set of m -tuples of pairwise distinct points in S for any $m \in \mathbb{N}$.

The following theorem explains the significance of the flexibility concept.

Theorem 1.1 ([1, Theorem 0.1]). *Let X be an affine algebraic variety of dimension ≥ 2 . Then the following conditions are equivalent.*

- (1) *The variety X is flexible;*
- (2) *the group $\text{SAut } X$ acts transitively on the smooth locus X_{reg} of X ;*
- (3) *the group $\text{SAut } X$ acts infinitely transitively on X_{reg} .*

Three classes of flexible affine varieties are described in [2], namely affine cones over flag varieties, non-degenerate toric varieties of dimension ≥ 2 , and suspensions over flexible varieties. Note that affine cones over del Pezzo varieties of degree ≥ 6 are toric, thereby they are flexible.

As for del Pezzo surfaces of degree ≤ 3 , the existence of at least one \mathbb{G}_a -action on an affine cone is still unknown, see [4], [6, Proposition 4.21]. In this paper we consider remaining cases of degree 4 and 5. In case of degree 5 we prove flexibility of affine cones corresponding to polarizations defined by arbitrary very ample divisors, whereas for degree 4 we prove flexibility only for certain very ample divisors, the anticanonical one included.

In the proof we use a construction from [6], which allows to associate a regular \mathbb{G}_a -action on an affine cone over a projective variety Y to every open cylindrical subset of Y of some special form. In Theorem 2.5 we provide a criterion of flexibility of an affine cone over a projective variety in terms of a transversal cover by such cylinders. We apply it to del Pezzo surfaces.

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2. FLEXIBILITY OF AFFINE CONES

Let Y be a projective variety and H be a very ample divisor on Y . A polarization of Y by H provides an embedding $Y \hookrightarrow \mathbb{P}^n$. Consider an affine cone $X = \text{AffCone}_H Y \subset \mathbb{A}^{n+1}$ corresponding to this embedding. There is a natural homothety action of the multiplicative group $\mathbb{G}_m = \mathbb{G}_m(\mathbb{K})$ on X . It defines a grading on the algebra $\mathbb{K}[X]$. A derivation on $\mathbb{K}[X]$ is called *homogeneous* if it sends homogeneous elements into homogeneous ones. A subset of all

homogeneous locally nilpotent derivations is denoted by $\text{HLND}(\mathbb{K}[X])$. The subfield of rational functions annihilated by all (resp. homogeneous) locally nilpotent derivations,

$$(1) \quad \text{FML}(X) = \bigcap_{\delta \in \text{LND}(X)} \text{Quot}(\ker \delta), \text{ resp. } \text{FML}^h(X) = \bigcap_{\delta \in \text{HLND}(X)} \text{Quot}(\ker \delta) \subset \mathbb{K}(X)$$

is called a *field Makar–Limanov invariant* (a *homogeneous field Makar–Limanov invariant* respectively). There is an obvious inclusion $\text{FML}(X) \subset \text{FML}^h(X)$. The invariant is said to be *trivial* if it equals \mathbb{K} .

Proposition 2.1 ([1, Proposition 5.1]). *The field Makar–Limanov invariant $\text{FML}(X)$ is trivial if and only if the group $\text{SAut } X$ acts on X with an open orbit.*

Definition 2.2 ([6, Definitions 3.5, 3.7]). Let H be a divisor on a variety Y . We say that an open subset $U \subset Y$ is a *cylinder* if $U \cong Z \times \mathbb{A}^1$, where Z is a smooth variety with $\text{Pic } Z = 0$. We say that a cylinder U is *H -polar* if $U = Y \setminus \text{supp } D$ for some effective divisor $D \in |dH|$, where $d > 0$.

Definition 2.3. We call a subset $W \subset Y$ *invariant* with respect to a cylinder $U = Z \times \mathbb{A}^1$ if $W \cap U = \pi_1^{-1}(\pi_1(W))$, where $\pi_1: U \rightarrow Z$ is the first projection of the direct product. In other words, every \mathbb{A}^1 -fiber of the cylinder is either contained in W or does not meet W .

Definition 2.4. We say that a variety Y is *transversally covered* by cylinders U_i , $i = 1, \dots, s$, if $Y = \bigcup U_i$ and there is no proper subset $W \subset Y$ invariant with respect to all U_i .

The following theorem gives a criterion of flexibility for the affine cone over a projective embedding $Y \hookrightarrow \mathbb{P}^n$ corresponding to the polarization by H .

Theorem 2.5. *If for some very ample divisor H on a normal projective variety Y there exists a transversal covering by H -polar cylinders, then the affine cone $X = \text{AffCone}_H Y$ is flexible.*

Proof. By [6, Theorem 3.9] to every covering cylinder there corresponds a \mathbb{G}_a -action on X . As follows from the explicit construction [6, Proposition 3.5], the orbits of this action map onto fibers of the cylinder under the cone projectivization, whereas the set of fixed points is a preimage of the cylinder complement.

Let us check similarly as in the proof of [6, Theorem 3.21] that the homogeneous field Makar–Limanov invariant $\text{FML}^{(h)}(X)$ is trivial. Indeed, let $h = \frac{f}{g}$ be a non-constant homogeneous rational function annihilated by all homogeneous locally nilpotent derivations on $\mathbb{K}[X]$, where $f, g \in \mathbb{K}[X] \setminus \{0\}$ are homogeneous regular functions of degrees d_1 and d_2 respectively. Then the divisor $\mathbb{V}_0(h) + \mathbb{V}_\infty(h)$ on X is invariant under the \mathbb{G}_a -actions corresponding to all covering cylinders. Thereby, $\mathbb{P}(\mathbb{V}_0(h)) \in |d_1 H|$ and $\mathbb{P}(\mathbb{V}_\infty(h)) \in |d_2 H|$ are effective Cartier divisors on Y , and the union of their supports is invariant with respect to all covering cylinders, which is a contradiction.

Thus $\text{FML}^{(h)}(X) = \mathbb{K}$. Hence also $\text{FML}(X) = \mathbb{K}$ and by Proposition 2.1 the group $\text{SAut}(X)$ acts on X with an open orbit. It remains to verify that the orbit complement is the only singular point of X , namely the origin. Since X is a cone, the open orbit is invariant under a \mathbb{G}_m -action by homotheties. This yields that the projectivization of the complement to the open orbit and the origin is a closed subset $W \subset Y$ which is invariant with respect to all cylinders of the transversal cover. This contradiction completes the proof. \square

3. DEL PEZZO SURFACE OF DEGREE 5

Let Y be a del Pezzo surface of degree 5. It is obtained by blowing up the projective plane \mathbb{P}^2 in four points P_1, \dots, P_4 , no three of which are collinear [7, Theorem IV.2.5]. Since the automorphism group of the projective plane acts transitively on such 4-tuples of points, such a surface is unique up to isomorphism.

Theorem 3.1. *Let H be an arbitrary very ample divisor on the del Pezzo surface Y of degree 5. Then the corresponding affine cone $\text{AffCone}_H Y$ is flexible.*

The proof proceeds in several steps, see Sections 3.1 and 3.2. We let E_i denote the exceptional divisor (i.e. the (-1) -curve), which is the preimage of the blown up point P_i . Let e_0 be the divisor class of a line, which contains none of the points P_i , and let e_i ($i = 1, \dots, 4$) be a divisor class of E_i . These classes generate a Picard group $\text{Pic } Y = \langle e_0, \dots, e_4 \rangle_{\mathbb{Z}} \cong \mathbb{Z}^5$. The intersection index defines a symmetric bilinear form on the Picard group such that the basis $\{e_0, \dots, e_4\}$ is orthogonal, $e_0^2 = 1$ and $e_i^2 = -1$. Exceptional divisor classes are e_i and $e_0 - e_i - e_j$ for distinct $i, j \neq 0$.

By the Nakai–Moishezon criterion [5, Theorem V.1.10] the closure of the ample cone $\text{Ample } Y$ is dual to the cone of effective divisors $\text{Eff } Y$. In case of a del Pezzo surface the cone $\text{Eff } Y$ is generated by exceptional divisors [3, Theorem 8.2.19]. Therefore, the ample cone is defined by inequalities

$$(2) \quad x_i > 0, \quad i = 0, \dots, 4,$$

$$(3) \quad x_0 + x_i + x_j > 0, \quad 0 \neq i \neq j \neq 0,$$

where $(x_0, \dots, x_4) \in \text{Pic } Y$. It has the following ten extremal rays

$$(4) \quad e_0, e_0 - e_j, 2e_0 - \sum_{i \neq 0} e_i, 2e_0 - \sum_{i \neq 0, j} e_i \quad j = 1, \dots, 4.$$

3.1. Cylinders. We have fixed above a blowing down $\varphi: Y \rightarrow \mathbb{P}^2$ of four pairwise disjoint (-1) -curves E_1, \dots, E_4 . Let $l_{ij} \subset \mathbb{P}^2$ be the line passing through the points P_i and P_j . Consider the open subset $U_1 = \varphi^{-1}(\mathbb{P}^2 \setminus (l_{12} \cup l_{34})) \subset Y$. This is a cylinder defined by a pencil of lines passing through the base point $\text{Bs}(U_1) = l_{12} \cap l_{34}$. We have $U_1 \cong \mathbb{A}_*^1 \times \mathbb{A}^1$. Similarly let $U_2 = \varphi^{-1}(\mathbb{P}^2 \setminus (l_{13} \cup l_{24}))$ and $U_3 = \varphi^{-1}(\mathbb{P}^2 \setminus (l_{14} \cup l_{23}))$, see fig. 1. Furthermore, consider the blowings down of other 4-tuples of non-intersecting (-1) -curves on Y . There are five of them as shown on fig. 2. For every blowing down we define three cylinders in a similar way.

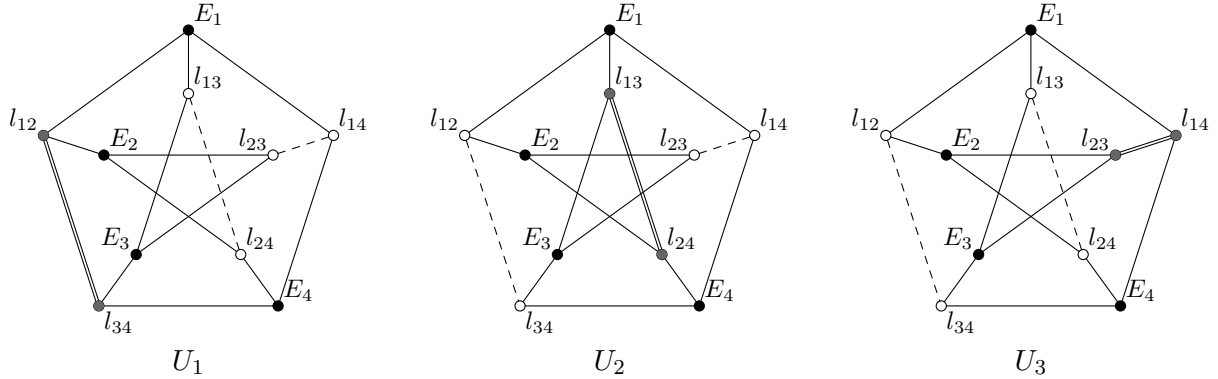


FIGURE 1. Cylinders on the incidence graph of (-1) -curves on the del Pezzo surface of degree 5. The gray and the black vertices correspond to (-1) -curves forming the complement to a cylinder. The dashed edges correspond to (-1) -curve intersections contained in the cylinder. The double edge corresponds to the base point of the cylinder.

Thus we have cylinders U_1, U_2, \dots, U_{15} as shown on Figures 1 and 2. It is easy to check that every intersection of (-1) -curves is contained in some cylinder, hence $\bigcup U_i = Y$. We claim that there is no proper subset $W \subset Y$, which is invariant with respect to all 15 cylinders. Assume on the contrary that there exists such a subset W . Let us fix an arbitrary point of W . It is covered by a fiber S of some cylinder, hence W contains S . Without loss of generality S is a fiber of U_1 . Then a line $l = \overline{\varphi(S)} \subset \mathbb{P}^2$ passes through the base point $\text{Bs}(U_1)$. Since the points $\text{Bs}(U_1)$, $\text{Bs}(U_2)$, and $\text{Bs}(U_3)$ do not lie on the same line, one of them does not belong to l . Suppose $\text{Bs}(U_2) \notin l$. Then the fiber S intersects almost every fiber of the cylinder U_2 , and W contains them. So, W is dense in Y . The complement $Y \setminus W$ is also invariant with respect to all cylinders, and by the same reason it is dense in Y , a contradiction.

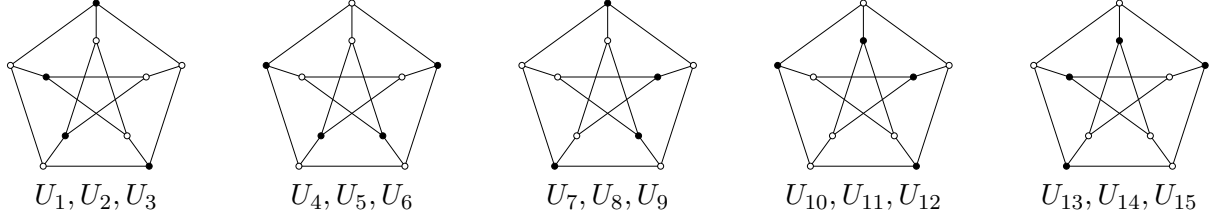


FIGURE 2. Black vertices correspond to 4-tuples of (-1) -curves. Every blowing down defines three cylinders similarly as on fig. 1.

3.2. Flexible polarizations. In this subsection we show that for any ample divisor H on Y all the 15 cylinders U_i are H -polar. Consider the set of effective divisors $\{\alpha_i E_i + \beta_1 l_{12} + \beta_3 l_{34} \mid \alpha_i, \beta_i > 0\}$ whose support is the complement to U_1 . The image of this set in the Picard group is an open cone C , whose extremal rays are $e_1, e_2, e_3, e_4, e_0 - e_1 - e_2$, and $e_0 - e_3 - e_4$. It is easy to check that the primitive vectors of the ample cone (4) can be expressed as linear combinations with non-negative rational coefficients of the primitive vectors of the cone C . Therefore the cylinder U_1 is H -polar for any ample divisor H . Similarly, the cylinders U_i are H -polar for any ample divisor H . Using Theorem 2.5 we obtain the assertion. Now Theorem 3.1 is proved.

4. DEL PEZZO SURFACES OF DEGREE 4

Every del Pezzo surface of degree 4 is isomorphic to a blowing up of a projective plane \mathbb{P}^2 in five points, where no three are collinear. Such surfaces form a two-parameter family.

By E_i we denote the (-1) -curve which is the preimages of the blown up point P_i . As before, let e_0 be the divisor class of a line which does not contain the blown up points, and e_i ($i = 1, \dots, 5$) be the divisor class of E_i . The tuple $\{e_0, \dots, e_5\}$ forms an orthogonal basis of the Picard group $\text{Pic } Y \cong \mathbb{Z}^6$, and $e_0^2 = 1, e_i^2 = -1$. The classes of (-1) -curves are $e_i, e_0 - e_i - e_j, 2e_0 - \sum_{k \neq 0} e_k$ for any pair of distinct indices $i, j \neq 0$. The ample cone is defined by inequalities

$$(5) \quad x_i > 0, \quad i = 0, \dots, 5,$$

$$(6) \quad x_0 + x_i + x_j > 0, \quad 0 \neq i \neq j \neq 0,$$

$$(7) \quad 2x_0 + x_1 + \dots + x_5 > 0,$$

where $(x_0, \dots, x_5) \in \text{Pic } Y$. Its extremal rays are

$$(8) \quad e_0, e_0 - e_j, 2e_0 - \sum_{k \neq 0, i} e_k, 2e_0 - \sum_{k \neq 0, i, j} e_k, \text{ and } 3e_0 - \sum_{k \neq 0} e_k - e_i$$

for any pair of distinct indices $i, j \in \{1, \dots, 5\}$.

4.1. Cylinders. Let us fix some (-1) -curve C_1 and consider the blowing down $\sigma: Y \rightarrow \mathbb{P}^2$ of the five (-1) -curves F_1, \dots, F_5 that meet C_1 , see fig. 3. This blowing down is well defined since the contracted divisors do not intersect. The image $\sigma(C_1)$ is a smooth conic c passing through the blown down points Q_1, \dots, Q_5 . Take an arbitrary line $l \subset \mathbb{P}^2$ which is tangent to c at a point different from Q_1, \dots, Q_5 . A conic pencil in \mathbb{P}^2 generated by divisors c and $2l$ determines a cylinder $U \cong \mathbb{A}_*^1 \times \mathbb{A}^1$ whose complement is the complete preimage of the support of the divisor $c + 2l$ on \mathbb{P}^2 . Denote by \mathcal{U}_{C_1} the family of all such cylinders in Y for all such tangents l . Note that $Y \setminus \bigcup_{U \in \mathcal{U}_{C_1}} U$ is a union of C_1 and the exceptional divisors F_i ($i = 1, \dots, 5$). Applying this construction to the (-1) -curves C_2, \dots, C_5 as shown on fig. 3, overall we obtain five cylinder families $\mathcal{U}_{C_1}, \dots, \mathcal{U}_{C_5}$. It is easy to see that their union covers Y .

Let W be a proper subset of Y which is invariant with respect to the cylinders of all families, and let $w \in W$ be an arbitrary point. We may suppose that w belongs to a cylinder of the family \mathcal{U}_{C_1} . Then the image $\sigma(W) \subset \mathbb{P}^2$ is invariant with respect to the cylinder family $\{\sigma(U) \mid U \in \mathcal{U}_{C_1}\}$. Note that every cylinder of this family is a complement to the conic c and its tangent line. It is well known that given a conic and two points outside it we can find a

conic passing through these two points and tangent to the given conic. Therefore, for almost every point $x \in \mathbb{P}^2 \setminus c$ there exists a fiber of some cylinder which contains x and $\sigma(w)$. Namely, x must not lie on the tangent line to c passing through $\sigma(w)$ as well as on the conics which are tangent to c at blown down points and contain $\sigma(w)$. Thus W is dense in Y . Similarly, $Y \setminus W$ is dense in Y , a contradiction. Finally, the families $\mathcal{U}_{C_1}, \dots, \mathcal{U}_{C_5}$ form a transversal cover of Y .

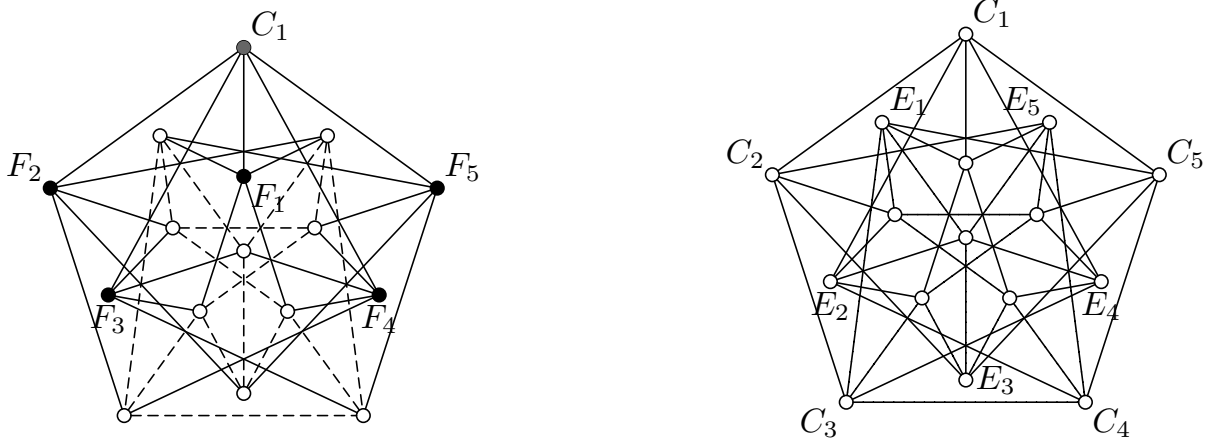


FIGURE 3. The incidence graph of (-1) -curves on a del Pezzo surface of degree 4. On the left the gray vertex corresponds to the conic preimage C_1 and black vertices correspond to the contracted (-1) -curves. The dashed edges correspond to (-1) -curve intersections contained in the cylinders of a family. Four other families corresponding to C_2, \dots, C_5 are obtained symmetrically by the graph rotations.

4.2. Flexible polarizations. Ample divisors H such that cylinders of the family \mathcal{U}_C are H -polar, form the image of the set $\{\alpha_1 F_1 + \dots + \alpha_5 F_5 + \alpha_6 C + \alpha_7 l \mid \alpha_i > 0\}$ in $\text{Pic } Y$. This set is an open cone which we denote by $\text{Ample}(C, Y)$. It does not depend on a choice of a tangent line l since it does not contain blown up points by definition. Then the set of such divisors H that cylinders in $\bigcup_i \mathcal{U}_{C_i}$ are H -polar is an open cone $\bigcap_i \text{Ample}(C_i, Y)$. A computation shows that it has exactly 72 extremal rays, which can be expressed as

$$\begin{array}{ll} e_0, & 9e_0 - 5e_{i_1} - e_{i_2} - 2e_{i_3} - 4e_{i_4} - 3e_{i_5}, \\ 4e_0 - 2e_{i_1} - 2e_{i_2} - e_{i_3} - e_{i_4} - e_{i_5}, & 9e_0 - 4e_{i_1} - 4e_{i_2} - 4e_{i_3} - 2e_{i_4} - 2e_{i_5}, \\ 5e_0 - 2e_{i_1} - 2e_{i_2} - e_{i_3} - 3e_{i_4} - e_{i_5}, & 11e_0 - 6e_{i_1} - 2e_{i_2} - 2e_{i_3} - 4e_{i_4} - 4e_{i_5}, \\ 5e_0 - 2e_{i_1} - 2e_{i_2} - 2e_{i_3} - 2e_{i_4}, & 11e_0 - 6e_{i_1} - 4e_{i_2} - 4e_{i_3} - 2e_{i_4} - 2e_{i_5}, \\ 5e_0 - 2e_{i_1} - 2e_{i_2} - 2e_{i_3} - 2e_{i_4} - 2e_{i_5}, & 11e_0 - 6e_{i_1} - 2e_{i_2} - 4e_{i_3} - 4e_{i_4} - 4e_{i_5}, \\ 6e_0 - 2e_{i_1} - 2e_{i_2} - 3e_{i_3} - e_{i_4} - 3e_{i_5}, & 11e_0 - 6e_{i_1} - 4e_{i_2} - 4e_{i_3} - 4e_{i_4} - 2e_{i_5}, \\ 7e_0 - 4e_{i_1} - 2e_{i_2} - 2e_{i_3} - 2e_{i_4} - 2e_{i_5}, & 15e_0 - 8e_{i_1} - 2e_{i_2} - 4e_{i_3} - 6e_{i_4} - 6e_{i_5}, \\ 9e_0 - 5e_{i_1} - 3e_{i_2} - 4e_{i_3} - 2e_{i_4} - 1e_{i_5}, & 15e_0 - 8e_{i_1} - 6e_{i_2} - 6e_{i_3} - 4e_{i_4} - 2e_{i_5}, \end{array}$$

where the tuple (i_1, \dots, i_5) runs over all cyclic permutations of $(1, 2, 3, 4, 5)$.

It is easy to see that the anticanonical divisor $(-K_Y)$ is contained in $\bigcap_i \text{Ample}(C_i, Y)$. Similarly to Theorem 3.1 we obtain the following result.

Theorem 4.1. *Let Y be a del Pezzo surface of degree 4, and H be a very ample divisor in the open cone $\bigcap_{i=1}^5 \text{Ample}(C_i, Y)$. Then the affine cone $\text{AffCone}_H Y$ is flexible. In particular, this holds for the anticanonical divisor $H = -K_Y$.*

We have identified a subcone of the ample cone such that the very ample divisors contained in this subcone define a flexible affine cone. However, this subcone is strictly contained in the ample cone. For example, the ample divisor class $8e_0 - 2e_1 - 4e_2 - e_3 - e_4 - 3e_5$ lies outside

of that subcone. Thus the flexibility problem for the affine cone over the polarization of a del Pezzo surface of degree 4 by any very ample divisor remains open.

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